Achievements and evaluation

The year was very productive, and major progress was made. We made progress in a variety of directions we had already planned, but also in new and unexpected ones.

Important results

1) Non-recurrent Fatou-Components.

In the field of complex dynamics, the problem of classifying Fatou components for various classes of holomorphic maps is a major challenging open problem. It has been known that any non-recurrent invariant attracting Fatou component for a polynomial automorphism of C^2 is necessarily biholomorphic to C^2. A striking and very surprising result, due to Bracci, Raissy and Stensønes [5] is that this does not continue to hold for holomorphic automorphisms of C^2: there exist holomorphic automorphisms of C^2, having invariant, non-recurrent, attracting Fatou components biholomorphic to $C^* \times C$.

2) The Muir-Suffridge conjecture.

A classical topic in Several Complex Variables is the question about boundary regularity of biholomorphisms between two domains. Famous examples are the classical Carathéodory extension theorem in one complex variable, and the celebrated Fefferman's extension theorem in Several Complex Variables, assuming high boundary regularity. Bracci and Gaussier [6] proved a remarkable regularity theorem with no assumptions of the boundary regularity of the target domain, solving the so called Muir-Suffridge conjecture.

3) Invariant Metrics.

Among other things, the understanding of invariant metrics is of fundamental importance in the study of complex dynamical systems, and of boundary regularity questions as mentioned above, e.g., the Bergman metric in the case of Fefferman. The metrics are almost never known explicitly, and so the asymptotic expansions of the metrics, a topic that have been studied extensively since the 70's, is of fundamental importance. Utilizing mapping techniques developed in relation to the Andersén-Lempert theory and Oka theory, Wold [18], and Bracci, Fornæss and Wold [7], have now obtained optimal expansions of the Carathéodory-Kobayashi metrics on strictly pseudoconvex domains, and even obtained
the "unbelievable" result that all metrics coincide in directions close to the tangent directions near the boundary.

4) Squeezing Functions.

A very recent notion in complex geometry is that of a squeezing function, an idea that was essentially introduced by S.T.Yau et ál in 2004. It should be thought of as a way of measuring how much a complex manifold resembles the unit ball observed from a given point, and is closely related to the study of invariant metrics as mentioned above. Through mapping techniques originating in the Oka theory, prior to this project, Fornæss and Wold have showed that any strictly pseudoconvex domain asymptotically resembles the unit ball in an optimal way. A question that has received a lot of attention, is whether the converse holds true. The most remarkable result in this direction was proved by A. Zimmer, who showed that this is indeed the case if the domain under consideration is convex, and has $C^{2,\alpha}$-smooth boundary, for $\alpha > 0$. Fornæss and Wold [8] proved the somewhat puzzling result that this breaks down if $\alpha = 0$. Furthermore, contrary to everyones expectation that ”optimal” behaviour of the squeezing function should imply that the domain under consideration should be ”nice” in a certain sense, Arosio, Fornæss, Shcherbina and Wold [3], constructed Cantor sets in P^1 of positive 2-dimensional Hausdorff measure, for which the complements have squeezing functions with ”optimal ” behaviour.

5) *The ∂-equation.*

Recent advances in the study of invariant metrics and squeezing functions, require machinery originating with the Oka- and Andersén-Lempert theory. This machinery requires solving the ∂-equation with precise boundary estimates, and has so far only worked for domains with so called Stein neighbourhood bases. To develop the theory on so called Worm-domains [7], (a sort of universal class of test domains where global properties tend to fail), which fail to have such bases, Bracci, Fornæss and Wold developed an intricate application of Hörmanders L2-techniques for solving the ∂-equation.
To study boundary regularity of the ∂-equation, it is of fundamental importance to have a detailed understanding and control of the geometry of boundaries of pseudoconvex domains. This is possible only in the real analytic case. The topic is well understood in complex dimension two, but it remains a major challenge to pass to higher dimensions. Progress on this has been made by Simon and Stensønes [17].

6) *Injective holomorphic mappings.*

One of the major open questions in the theory of injective holomorphic mappings, among other things related to Loewner theory in higher dimensions, is whether there is a general approximation theory for holomorphic maps with local or global injectivety properties. An ultimate goal would be to radically supersede the Andersén-Lempert theory. As a step in this direction, Forstneric [9] has showed that any continuous map $f : S \to X$ from a Stein manifold S to an Oka manifold X, is homotopic to a strongly dominating map.

7) *The Oka Principle.*

One of the central problems in Oka theory is to determine to what extent Oka properties are persistent with respect to modifications of the space under consideration. Ultimately, one would like to classify Oka manifolds. Kutzschebauch, Kaliman and Truong [12] made serious progress on the topic, proving that certain blow-ups of a certain class of manifolds are algebraically sub-elliptic, and thereby Oka manifolds.

An important problem in analysis and geometry is to describe the rough shape of various mapping spaces. Answering such a question typically amounts to proving a homotopy principle, to the effect that analytic solutions exist providing topological solutions exists. This is the core of Oka theory. In the paper [10], Forstneric and Larússon proved that the space of Legendrian immersions of a given curve in C^{2n+1} is weakly homotopy equivalent to the space of continuous maps from the same curve into B^{4n-1}.

To avoid going into technicalities we note that Kutzschebauch, Larússon and Schwarz [11] have proved an equivariant parametric Oka principle for bundles of homogenous spaces.

8) *Trancendental Hénon Maps.*

Perhaps the most studied class of maps in higher dimensional dynamical systems, is the class of Hénon maps. This is the class of polynomial automorphisms of C^2 that gives rise to ”interesting” dynamical behaviour. On the one hand, however, there is no reason to believe that all interesting natural phenomena are described by polynomial equations. On the other hand, if we consider arbitrary classes of maps, there is not enough structure to expect to get results. Arosio, Benini, Fornæss and Peters [2], have introduced and taken the first step towards studying so-called transcendental Hénon maps, which generalize classical Hénon maps. They proved the existence of oscillating wandering domains, escaping wandering domains and Baker domains.

9) *Polynomial Hulls.*

A key problem when considering approximation theory in Several Complex Variables, in particular in connection with Oka theory and Andersén-Lempert theory, is to understand so called polynomial convexity. In particular, it is important to understand when certain sets, or isotopies of sets, ”generically” are polynomially convex. On the other hand, if a set is not polynomially convex, it is important to understand what exactly it is that prevents this. It has been understood that if the sets in question are certain compact totally real Riemann surfaces in C^n, and under certain dimension assumptions, the culprit is always the presence of analytic structure in the maximal ideal space of the associated algebra. Opposed to this, Arosio and Wold [4] have proved that any compact C^∞ manifold of dimension d admits totally real embedding into $C^{3d/2}$ with a non-trivial polynomial hull without complex structure.

10) *Infinite Dimensional Complex Analysis.*

Berndtsson, Cordero Erauskin, Klartag, and Rubinstein have proposed a generalisation of the Legendre transform to Kähler manifolds, and showed that this gives rise to local isometries, with respect to the Mabuchi metric, near highly regular points. Lempert [13] characterised such local isometries between spaces of Kähler potentials, and gave existence and uniqueness properties for such isometries. Furthermore, he showed [14] that conversely, the existence of a local isometry implies this high regularity. His article [15] is devoted to collecting results on infinite dimensional Riemannian geometry that are needed in the two other articles.

11) *Anosov Conjecture.*

This topic fits into the realm of continuous dynamical systems. There is a well known conjecture of Anosov, stating that for a generic foliation by Riemann surfaces on complex projective space of dimension two, all but countably many leaves of the foliation are conformal to the unit disk. Continuing a project started prior to the CAS year, Sibony (twice visitor at CAS) and Wold [16] completed a project where they have introduced ergodic theory in this setting, showing in some specific sense, that the conjecture is either true, or very far from being true.

Work in progress

In addition to the work mentioned above, several projects were started during the year, but has not yet reached the maturity to be presented in any detail. We mention a few of these.

1) John Erik Fornæss, Nessim Sibony and Erlend F. Wold. Related to the problem of understanding the geometry of real analytic pseudoconvex boundaries, it seems to be an almost unexplored direction to study pseudoconvexity properties of real analytic CR-manifolds of higher co-dimension. It is not even clear what a pseudoconvex CR-manifold should be in general. In this direction, Fornæss, Sibony (twice a visitor at CAS) and Wold have started a project, also studying related topics like plurisubharmonic approximations, the Dirichlet-Bremerman problem, and an ultimate long term goal is to solve the Levi problem for singular spaces.

2) Björn Ivarsson, Frank Kutzschebauch and Erik Løw. Ivarsson, Kutzschebauch and Løw have worked persistently on the so-called Symplectic Gromov Vaserstein problem throughout the whole CAS year. The problem was solved in general for continuous matrices and in the (4×4)-case for holomorphic matrices. Passing to larger matrices seems a highly non-trivial long-term project due to an exceptional degree of computational complexity.

3) László Lempert, Frank Kutzschebauch and Erik Løw. Lempert, Kutszchebauch and Løw started a collaboration on automorphisms of large complex manifolds, and on holomorphic maps into such manifolds. An ultimate goal is to understand whether there exist so-called Fatou-Bieberbach domains in $C^*\times C^*$. There has not been obtained definitive results yet.

4) Finnur Larússon and Leandro Arosio. Larússon started a project on algebraic Oka theory with T. Truong during the workshop in Jevnaker in October 2016. The project continued after the end of the CAS year, and by now has been completed and submitted. With Arosio, he also developed a detailed plan for a project on the interface between Oka theory and dynamical systems, a project that will commence late 2017.

5) Fornæss, Trybula, Wold. To attack the persistent problem of deciding whether any open Riemann surface is a closed complex subvariety in C^2, Fornæss, Trybula (visitor at CAS) and Wold have introduced and started to study the injective Kobayashi metric on two dimensional complex manifolds. This metric seems to be highly relevant, as it turns out that it governs the growth rate of approximations by injective holomorphic maps.

Publications

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[3] Arosio, L., Fornæss, J. E., Shcherbina, N. and Wold, E. F. 2017. ”Squeezing functions and Cantor Sets.”

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[5] Bracci, F., Raissy, J. and Stensønes, B. 2017. ”Existence of a non-recurrent attracting Fatou component for automorphisms of C^2.” arXiv: 1703.08423.

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