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Non Cooperative Differential Games

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1 Finite Games

The analysis of decision making is a primary goal of the mathematical theory of games. This discipline originated with the classic book by J. von Neumann and O. Morgestern [11], and seminal contributions by J. Nash [10] and R. Isaacs [9].

In the simplest setting, one considers two players: A and B, who can choose among a number of possible options. Say, player A can choose within the set of strategies $\mathcal{S}_A = \{a_1, a_2, \ldots, a_m\}$, while $\mathcal{S}_B = \{b_1, b_2, \ldots, b_n\}$ is the set of strategies available to player B. It is assumed that the choices are made simultaneously: a player cannot wait and see what the other does, before making his decision. The game is now described by two payoff functions: $P_A(a_i, b_j)$ and $P_B(a_i, b_j)$. These are the payoffs achieved by A and B respectively, if the first player adopts the strategy a_i and the second chooses b_j. The games studied in the earlier mathematical literature were "zero sum", in the sense that

$$P_A(a_i, b_j) = -P_B(a_i, b_j) \tag{1.1}$$

for every a_i, b_j. In this case, the two players are strictly antagonistic: the amount gained by the first one coincides with what the second one loses. A familiar example is the so-called "rock-paper-scissors" game. Here each player has the same three options, namely $\mathcal{S}_A = \mathcal{S}_B = \{r, p, s\}$. The payoffs for the two players are displayed in Figure 1, left.

As argued in [11], here the best strategy for each player is to randomly choose one of the three options r, p, s with equal probabilities: $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$. Since this game is symmetric, there can be no way for any player to forcibly achieve a positive payoff. However, the above random strategy at least guarantees against any loss, on average.

A different approach is required in the case of non-zero-sum games, where the condition (1.1) fails. Here the actions taken by the players, and the eventual outcome of the game, may also depend on their ability and willingness to cooperate.

A classic example is the so-called "prisoner's dilemma". Two prisoners are interrogated. Each one has two options: either confess and accuse the other prisoner, or not confess. If he confesses, the police rewards him by reducing his sentence. The expected number of years in jail faced by the two prisoners is displayed in Figure 1, right. The upper triangles contain the

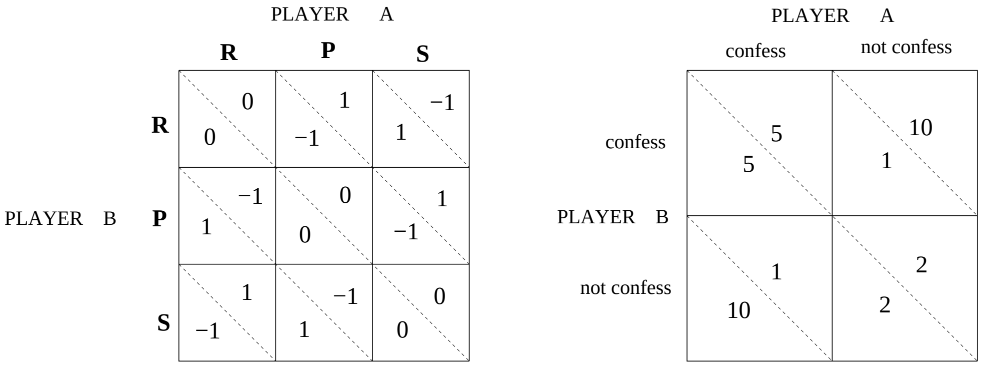

Figure 1: Left: the payoffs for the "rock-paper-scissors" game. Right: the penalty functions for the "prisoner's dilemma".

number of years in jail for player A, while the lower triangles refer to player B. None of the prisoners, while interrogated, knows about the behavior of the other.

Taking the side of player A, one could argue as follows. If player B confesses, my two options result in either 5 o 10 years in jail, hence confessing is the best choice. On the other hand, if player B does not confess, then my two options result in either 1 or 2 years in jail. Again, confessing is the best choice. Since the player B can argue exactly in the same way, the outcome of the game is that both players confess, and get a 5 years sentence. In a sense, this is paradoxical because an entirely rational argument results in the worst possible outcome: the total number of years in jail for the two prisoners is maximal. If they cooperated, they could both have achieved a better outcome of only 2 years in jail.

The previous example illustrates the concept of Nash non-cooperative equilibrium. Namely, we say that (a_{i^*}, b_{j^*}) is an equilibrium solution to the game with strategies $\mathcal{S}_A$, S_B and payoffs P_A, P_B if the following holds. If player A changes his strategy from a_{i^*} to any other strategy $a_i \in \mathcal{S}_A$, then his payoff will not increase:

$$P_A(a_i, b_{j^*}) \leq P_A(a_{i^*}, b_{j^*}) \qquad \text{for all } a_i \in \mathcal{S}_A\,.$$

Similarly, player B cannot increase his payoff by switching to any other strategy:

$$P_B(a_{i^*}, b_j) \leq P_B(a_{i^*}, b_{j^*}) \qquad \text{for all } b_j \in \mathcal{S}_B\,.$$

In other words, if a player single-handedly changes his strategy, his payoff will not increase. Notice that this concept of non-cooperative solution models a situation where the players do not communicate and cannot strike deals with each other. In particular, here we do not consider the possibility that the two players simultaneously change their strategy, increasing both payoffs.

2 Differential Games

The previous section was concerned with games where each player is allowed a one-time choice, from a finite set of options. Many applications, on the other hand, lead to games in continuous time. Each player must here devise a course of action implemented over a whole interval of time, possibly of infinite duration.

For example, think of a rocket which is launched to destroy an enemy plane. Throughout its flight, the rocket must be steered toward the plane, adjusting its trajectory in real time, to hit the target. Similarly, the airplane pilot will maneuver trying to evade the incoming missile. Pursuit games of this type motivated the theory of differential games [9, 8]. Other amusing examples of differential games studied in the mathematical literature are the "homicidal chauffeur", where a car tries to hit a pedestrian on a large empty parking lot, or the "lion and man" game, where a lion chases a man with equal or lower speed, in the ancient roman Colosseum. Pursuit games are a special type of zero-sum games. The pursuer's strategy aims at minimizing the time of capture. On the other hand, the quarry seeks to maximize the capture time; better yet, he would like this time to be $+\infty$, which means that capture never occurs.

A general formulation of these differential games is as follows. We consider a (possibly vector valued) variable x describing the state of the system. For example, in a pursuit game the various components of the vector x could describe the instantaneous position of the chaser and of the quarry. This state will of course change in time. Its evolution is governed by a differential equation of the form

$$\frac{d}{dt}x(t) = G(x(t), u_1(t), u_2(t)) \,. \tag{2.1}$$

Given an initial state $x(t_0) = x_0$, the future evolution of the system is thus determined by the inputs $u_1(t)$, $u_2(t)$ of the two players. Together with (2.1) we consider two payoff functions J_1, J_2. For example, in the case of games with a finite terminal time T, one can define

$$J_i \;=\; \psi_i(x(T)) - \int_{t_0}^{T} L_i(x(t), u_1(t), u_2(t))\, dt \qquad\qquad i = 1, 2\,. \tag{2.2}$$

Here ψ_i denotes the terminal payoff for the i-th player, while L_i denotes a running cost.

Alternatively, one can consider games with infinite time horizon. In this case, it is natural to insert an exponential factor, discounting the value of gains far into the future. For some discount factor $\rho > 0$, we thus define

$$J_i \;=\; \int_{t_0}^{T} e^{-\rho t} \cdot \Phi_i(x(t), u_1(t), u_2(t))\, dt \qquad\qquad i = 1, 2\,. \tag{2.3}$$

A possible approach to these type of differential games relies on the analysis of the *value functions* V_i. Roughly speaking, for a given initial state $x(0) = y$, the value $V_i(y)$ is the payoff which can be achieved by the i-th player, if he plays optimally.

At this stage, a natural assumption is that the strategies adopted by the two players have feedback form: $u_i = u_i^*(x)$; namely, they depend only on the current state of the system and not on the past history. For a Nash non-cooperative solution in feedback form, one can show that the value functions V_i satisfy a system of (Hamilton-Jacobi-Bellman) partial differential equations, derived from the principle of dynamic programming [7]. For the infinite horizon game (2.1), (2.3), these equations take the form

$$\rho V_i(x) = H_i(x,\, \nabla V_1(x),\, \nabla V_2(x)) \qquad\qquad i = 1, 2\,. \tag{2.4}$$

In turn, from the spatial gradients ∇V_i of these value functions one can recover the feedback controls $u_1^*(x)$, $u_2^*(x)$ for the two players.

For zero-sum games, one has $V_2(x) = -V_1(x)$, and the system (2.4) reduces to one single scalar equation. In spite of the difficulties caused by the non-linearity of the equation and by the lack of regularity of the solutions, an extensive mathematical theory is now available. This is largely based on the concept of *viscosity solution* [2].

In economic theory, it is common to consider a situation where there are several agents, producers and consumers, operating in the same environment. Each player wishes to maximize his own payoff or utility function. Moreover, the overall state of the economy is affected by the combined actions of all players. This is definitely *not* a zero-sum game, and a PDE approach must deal with a system of several equations, of the form (2.4). In general, this system is difficult to study, because the functions H_i are highly nonlinear. Moreover, as shown by the analysis in [5, 6], the problem can often be "ill-posed". Namely, a small perturbation of the data may cause a large change in the solution. At present, the main body of mathematical theory is concerned with *linear-quadratic* games [3], where G is a linear function of its arguments x, u_1, u_2, and the payoff functions ψ_1, L_i, Φ_i in (2.2) or (2.3) are quadratic polynomials. In this case, one can find a solution to the corresponding system of PDEs within the class of quadratic polynomials.

For a new approach, relating a general non-linear differential game to a problem of optimal control by a "homotopy" technique, we refer to [4].

3 Concluding remarks

In this note we outlined an approach to differential games, based on the analysis of the PDEs describing the value functions. Various other approaches can be considered, also motivated by different modelling assumptions. We remark that an appropriate mathematical model should take into account: (i) the information available to each player, and his ability to use this information to compute the optimal strategy, and (ii) the ability of the players to communicate with each other, and their willingness to cooperate. A theoretical analysis of these games can have multiple goals:

• determine which is the best course of action for a given player,

• predict players' behavior in real life situations,

• understand how the "rules of the game" can be modified, so that the eventual outcome of the game is "favorable" from the higher standpoint of a collectivity.

Apart form the case of two-players, zero-sum games, the mathematical theory for decision models involving several agents is still largely undeveloped. It is expected that these problems will provide a rich field for investigation in years to come.

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