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G.P. CHAPMAN AND N.H. RISEBRO

The partitioning of networks under simultaneous cooperation and competition

Introduction

The idea of balance in a network has been around for sometime. There are theorems which describe the characteristics of balanced networks in papers dating from the 1950s [4, 1]. Although these papers hint at forces which might cause a network to evolve towards a balanced state (which may involve a partition of the network) they do not examine mechanisms that might stimulate such evolution, nor the sequence of changes that the evolution may take. More recent work has examined network evolution (see [6], for a wide-ranging survey) but this work does not include the idea of "balance" as a driver of evolution.

Professor Emeritus G. Chapman
grahampchapman@googlemail.com
CAS fellow 2008/2009

Professor N.H. Risebro
Centre of mathematics
for Applications
Oslo, Norway
nilshr@math.uio.no
CAS fellow 2008/2009

In this paper we examine the idea of balance (hereafter referred to as "stability") in relation to network evolution. Cartwright and Harary [1] attribute the idea of balance to a psychologist [5], who reflected on how people liked other people who liked the same things, or liked people who liked their friends. An older appreciation of the same problem stems from the treatise on statecraft, the Arthashastra, written during the reign of the Indian Emperor Kautilya c. 250 BCE , probably by Kanakya, one of his senior advisors. In it is written "my enemy's enemy is my friend". This simple phrase encapsulates the driving force for the solution of a triangle of relationships, in which the third relationship is conditioned by the previous two.

There are four possible configurations of positive and negative edges for a single triangle [2, Figure 1]. If these plusses and minuses multiplied, then overall the value of these four possible sets of values becomes $+$, $-$, $+$, $-$. The two which come out positive are called stable patterns, and the two which come out negative are called unstable. In our case, using three countries as an example, in Figure 1a Pakistan, India and the USA all like each other, all cooperate, so the pattern is stable. In Figure 1b Pakistan and India dislike each other intensely. The USA tries to like both, but at some stage India may turn to the USA and say "You are helping my enemy, you either help me against my enemy, or I will reject your friendship." Pakistan will do the same. In essence, if the Indians and Pakistanis do not become good friends, the Americans will feel pressure to choose one or the other. The last state, Figure 1d is the unstable state of three negatives. History is riddled with examples of this, and the outcome: Churchill forming a pact with the "Devil himself" - Stalin - to defeat Hitler.

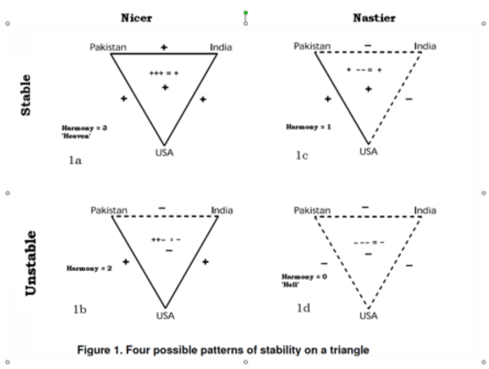

Figure 1

Here we propose two measures to indicate the state of the network in terms of "stability" and "harmony". Let H (the harmony) denote the number of positive edges in the network, and let S denote the number of positive edges. It is quite evident from Figure that a less harmonious pattern (c) may be more stable than a more harmonious one (b). Thus it becomes significant whether increasing H or S drives evolutionary change. Consider a network of N nodes, where $N \geq 3$. There are $E = N(N-1)/2$ edges, and there are $T = N(N-1)(N-2)/6$ triangles.

We can characterize a state of a network by its position in (H,S) space. Figure **??** shows the (H,S) space for a network of 6 nodes, which has 15 triangles and 20 edges. The small spheres indicate feasible states in the phase space so that (10,20) is possible, whereas $(9,10)$ is not. The lines represent changes by reversing the sign of a single edge. So it is possible to go from $(3,10)$ by changing a single edge to any of the four following states: $(2,8)$, $(4,8)$, $(4,10)$, $(4,12)$.

Consider the 6 node configuration in the left part of Figure 2, which is at $(5,12)$ in (H,S) space, see Figure **??**. How will we evolve this structure? We suggest that an iterative procedure is followed, in which the most unstable edge has its sign reversed at each iteration. The iteration is terminated when each edge borders on more stable triangles than unstable ones. The right configuration of Figure 2 shows the end result of this iteration. Here all triangles are stable. The evolution in (H,S) space is shown by a bold line in Figure 2. We

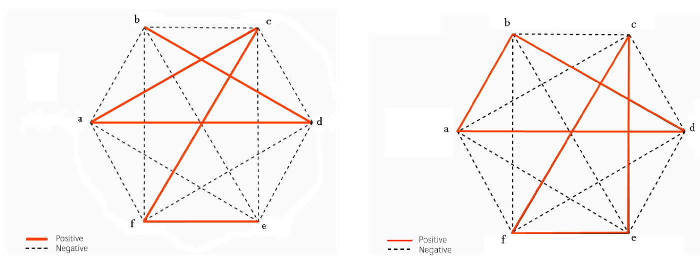

Figure 2. Left: initial configuration, right: final configuration.

measure the stability of an edge as the number of positive triangles minus the number of negative triangles bordering on that edge. In the graph on the left of Figure 2 there are two edges which are equally unstable, the edge between *ab* and the edge *ce*. If we reverse the sign of *ab*, then we move one step up the bold line in Figure **??**. This process is repeated until we reach the configuration to the right of Figure 2. The structure has become disconnected and arranged in two competing alliances. Put another way, what was competition and cooperation at an individual level becomes cooperation within alliances and competition between. All connections within each alliance are positive, and all connections between alliances are negative (competition).

This procedure is deterministic, but it contains within it an arbitrary criterion for choosing between the two equally unstable edges *ab* and *ce*. Choosing to change *ce* first results in a different sequence of change, but the same edges are changed, the same path is followed in the phase space Figure **??**, and the same partition results. We comment further on this in [3].

Mathematical formalism

Now we turn to a mathematical description of the evolution of this type of network. A network with N nodes can be described by a symmetric $N \times N$ matrix M whose entries m_{ij} are

$$m_{i,j} = \begin{cases} 0 & i = j, \\ 1 & \text{if } i \text{ is allied with } j, \\ -1 & \text{if } i \text{ is not allied with } j. \end{cases}$$

Using this formalism, we see that the triangle spanned by nodes i, j and k is stable if and only if $m_{ij}m_{jk}m_{ki} = 1$. If σ_{ij} denotes the stability of the edge between the nodes i and j (measured by the number of positive minus the number of negative triangles of which the edge ij is a part), then we can compute this by

$$\sigma_{ij} = \frac{1}{2}\sum_{k=1}^{N} m_{ij}m_{ik}m_{kj} = \frac{m_{ij}}{2}\left(M^2\right)_{ij}$$

The factor $1/2$ stems from the fact that we are counting "*both*" the edges ij and ji. The evolution of the network is described by the algorithm: Letting μ

```
σ_ij = ½ m_ij (M²)_ij
[ij] = argmin_kl {σ_kl}
while σ_ij < 0 do
   m_ij ← −m_ij
   σ_ij = ½ m_ij (M²)_ij
   [ij] =
   argmin_kl {σ_kl}
end while
```

Figure 3. The evolution of a network.

denote the total number of stable triangles minus the total number of negative triangles, and $T = N(N-1)(N-2)/6$ the total number of triangles, we have that

$$\mu = \frac{1}{3\cdot 2}\sum_{ij}\sigma_{ij},$$

where the factor 6 in the denominator arises since we are counting each triangle three times, one time for each edge, and each edge twice. If S denotes the number of stable triangles and Q the number of unstable triangles, using the obvious relations

$$S + Q = T, \quad S - Q = \mu,$$

yields that the number of stable triangles is given by

$$S = \frac{1}{2}(\mu + T).$$

Lemma 1. *Under each iteration of the algorithm in Figure* **??**, *the number of stable triangles increases.*

Proof. Let S and S' be the number of stable triangles before and after one iteration of the algorithm. Assume that the edge changed by the iteration is between the nodes p and q, so that $m'_{pg} = -m_{pq}$. Then also $\sigma'_{pq} = -\sigma_{pq} > 0$. Now

$$
\begin{aligned}
S' - S &= \frac{1}{2}\left(\mu' - \mu\right) = \frac{1}{12}\sum_{ij} \sigma'_{ij} - \sigma_{ij} \\
&= \sigma'_{pq} - \sigma_{pq} \\
&= 2\sigma'_{pq} > 0.
\end{aligned}
$$

□

This means that the algorithm terminates in a finite number of steps. In these terminal states, each edge are part of more stable triangles than unstable triangles. We call these terminal states *fixed states*. This is related to the notion of a balanced network, see [1]. A balanced network is a network for which all cycles are positive. This is equivalent to a network where all triangles are stable, so obviously a balanced network is in a fixed state.

Regarding fixed states we make the following conjecture:

If a network is in a fixed state, it does not contain any unstable triangles of type 1b (see Figure).

Unfortunately, we are not able to prove this at the moment. (But by brute force we have demonstrated the conjecture for any network where $N < 9$.) However, assuming that the conjecture holds, we can characterize the fixed states. These are of four types:

1. Heaven. This is a network for which $m_{ij} = 1$ for all i and j.

2. Two coalitions. The network can be divided into two groups, M_1 and M_2, for which

$$
m_{ij} = \begin{cases} 1 & \text{if } i \in M_1 \text{ and } j \in M_1, \text{ or } i \in M_2 \text{ and } j \in M_2, \\ -1 & \text{otherwise.} \end{cases}
$$

3. Three coalitions. The network can be divided into three groups, M_1, M_2 and M_3, such that M_l consists of n_l nodes and

$$
m_{ij} = \begin{cases} 1 & \text{if } (i,j) \in M_l \times M_l, \\ -1 & \text{otherwise.} \end{cases}
$$

If we number the groups such that $n_1 \leq n_2 \leq n_3$, then

$$
n_1 + n_2 > n_3 + 2.
$$

Case (1) and (2) are obviously fixed states, since every triangle is stable. To show that case (3) is stable, first note that all triangles with two nodes in one group are stable, and the only unstable triangles are those of type 1d, with one node from each group. Consider one edge ij where $i \in M_1$ and $j \in M_2$. This edge borders on n_3 unstable triangles of type 1d (for which the third node is in M_3, and $n_1 - 1 + n_2 - 1$ stable triangles of type 1c (for which the third node is in M_1 or M_2. Thus

$$
n_1 + n_2 - 2 > n_3.
$$

Similarly we get

$$n_2 + n_3 - 2 > n_1, \text{ and } n_1 + n_3 - 2 > n_2.$$

Adding these equations we find that $N > 6$.

To show that we cannot have any other fixed states, note that our conjecture implies that any fixed state consists of isolated coalitions. Assume that there are four such groups, $M_1, \dots, M_4$ each consisting of $n_1, \dots, n_4$ nodes. Consider an edge ij where $i \in M_1$ and $j \in M_2$. This edge will be part of $n_1 + n_2 - 2$ stable triangles and of $n_3 + n_4$ unstable ones. Thus

$$n_1 + n_2 - 2 > n_3 + n_4.$$

By symmetry,

$$\begin{aligned} n_1 + n_3 - 2 &> n_2 + n_4, \\ n_1 + n_4 - 2 &> n_2 + n_3, \\ n_3 + n_4 - 2 &> n_1 + n_2, \\ n_2 + n_3 - 2 &> n_1 + n_4. \end{aligned}$$

Therefore

$$n_1 + n_2 - 2 > n_3 + n_4 > n_1 + n_2 + 2 \;\Rightarrow\; -2 > 2.$$

Therefore such configurations cannot be fixed.

Note that if (1) or (2) holds, then the network is balanced, whereas networks with three interconnected groups are not balanced.

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