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Big Problems in Mathematics: Solved and Unsolved[1]

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There is a slight and unintended double meaning in the title: It is *not* a problem *for* mathematics that there are big problems *in* mathematics. Rather to the contrary: Mathematics has advanced through some of its big problems. We will here review some of these problems[2].

Let us start with Greek mathematics: Euclid's (approximately 300 BC) "Elements" describes geometry in the plane, and for first time in human history the axiomatic system is introduced: Certain concepts are left undefined[3], e.g., points and lines, and their properties are described by axioms or postulates that are taken for granted. An example is the following: Through two points one can draw one unique line. From the axioms, using the rules of logic, one can deduce valid statements called theorems. An example is the classical result that the angles of a triangle sum to 180 degrees, or the Pythagorean theorem, saying that the square of the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides. These results are true mathematical statements, but their relevance to nature depends on how well the abstract properties (axioms) for the undefined concepts (points and lines) reflect properties of "real" points and lines. The idea of an axiomatic system was bold and revolutionary. The choice of axioms has to be judicious: The system should be *independent* (i.e., it should not be possible to derive one axiom from another), it should be *consistent* (i.e., it should not be possible to use some axioms to prove that the angles in a triangle sum to 180 degrees, and another set of axioms to show that the sum equals 150 degrees), and finally the system should be *complete* (i.e., if you claim that the angles in any triangle sum to 180 degrees, it should be possible to prove or disprove this statement). We will later return to these issues in more detail. One postulate caused much concern to Euclid, namely the infamous parallel postulate. Rephrased it can be stated as follows: Given a line and a point not on

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[1] The portraits in the present article are from Wikipedia. Helpful comments from Harald Hanche-Olsen and the referee are well appreciated.

[2] The proof of Fermat's theorem is not discussed here due to lack of space and the extensive coverage that exists elsewhere.

[3] They are only loosely defined in the introduction. A "point" is "that which has no part", and a "line" is "a breadthless length". However, these properties are never used in the proofs.}

the line, it is possible to draw exactly one line through the point and parallel to the given line.

Euclid

This was not an obvious statement to Euclid, certainly less evident than the other axioms, but it was needed for the axiomatic system. However, all attempts to falsify it or to derive it from the other axioms failed, and this was the state of affairs for about 2000 years. The discovery of what is now called *non-Euclidean geometry* was a revolution in mathematics as well as in physics. What the Greeks had not been able to do, was to detach the mathematical properties of points and lines from "real" points and lines, and rather regard the axiomatic system as a game with certain rules. Once you do that, you can look for models different from those of planar geometry in which "points" and "lines" correspond to different objects than the physical points and lines, and where the parallel postulate is not true, either because there exist more than one parallel line or none. If one can find such a model, the parallel postulate cannot be derived from the other axioms. And such models were closer than one would think: Instead of considering the geometry on a small scale, one can consider the geometry on the surface of the Earth, and interpret lines as great circles, which satisfy the axioms, but do intersect[4]. Once the idea emerges to detach oneself from points and lines in the plane, one can come up with different axiomatic systems suitable to describe other geometries. The physicists have gotten a new tool to describe different physical systems, and mathematicians have new systems to analyze. As simple as this sounds today, it took more than two millennia to reach this understanding.

Let us now turn to another topic, that of solving polynomial equations. We all know how to solve first degree equations ($x+a=0$ has solution $x=-a$) and

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[4] We need to identify antipodal points, that is, points at opposite ends of a diameter, e.g., the North and South Pole. In this geometry there are no parallel lines.

second degree equations ($x^2+bx+c=0$ has solutions $x=\frac{1}{2}(-b\pm\sqrt{b^2-4c})$), but to get beyond this is harder. Indeed through the Renaissance, mathematicians could solve third and fourth degree equations[5]. The quest took place as challenges: One mathematician challenged another by revealing that he could indeed solve third degree equations without explaining the method. However, in spite of numerous attempts by many eminent mathematicians, no one could solve the quintic. This remained unsolved for several centuries until the Norwegian mathematician Niels Henrik Abel (1802–29) proved that there cannot exist a formula involving only roots and algebraic operations that solves general polynomial equations of degree five and higher.

Niels Henrik Abel

This is a statement on a considerably higher intellectual level: Abel proved that however hard you try, it is impossible to come up with such a formula for the solutions of general polynomial equations.

Abel's theorem, together with the theory of the French mathematician Évariste Galois (another genius who died young), was the starting point of what is now called group theory. This theory underpins for instance the theory of elementary particle physics, internet security, and most disciplines in mathematics.

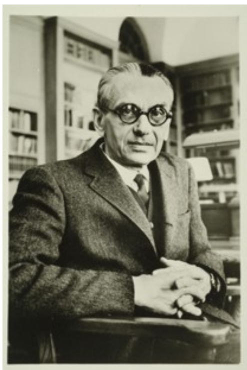
Let us now turn to the year 1900 and the International Congress of Mathematicians in Paris. The German mathematician David Hilbert (1862–1943) presented 23 open problems, which he considered the most important problems for mathematics for 20th century. It was probably the last time a single mathematician could claim with some conviction and authority to understand all branches of mathematics.

[5] Two remarks are appropriate here: The formulas that we talk about here are never used for numerical computations of the solutions; for that purpose there are general method applicable to equations of any degree. Furthermore, the interest in the formulas *per se* has declined.

David Hilbert

Although Hilbert could not foresee many important developments, for instance the level of abstraction in modern mathematics and the development of computers, it is fair to say that Hilbert's problem have been very important. Some problems were “too simple” and were solved shortly thereafter; some problems were too vague to say whether they have been solved at all, other problems simply appeared to be less interesting than Hilbert anticipated. But the majority of problems turned out to be central, and several problems remain unsolved. Let us look at some selected problems.

The second problem asked for a proof of the consistency of arithmetic, which in some sense is a return to the question asked in Greek mathematics: How can one be sure that one avoids contradictions? To put it bluntly: Can one be sure that no one can prove that $1+1=3$? This question, together with the other fundamental problem concerning the axiomatic method, was answered by the Austrian mathematician Kurt Gödel (1906–78).



Kurt Gödel

He proved the following fundamental and revolutionary result: One cannot prove the consistency of arithmetic in the way envisaged by HIlbert! And this is not all, he went on to prove that the axiomatic system can never be complete. One can always make mathematical statements within the theory that are either true or false, but which cannot be proved within the theory. One can say that the axiomatic system is not “strong enough” to decide if the statement is true or not. The problem is fundamental in the following sense: Independent of how one extends the axiomatic system it will still possess statements that can neither be proven nor disproved. This is rather abstract, but not more so than Hilbert's first problem turned out to be exactly an example of the above situation! His first problem asks if there exists a subset of the real numbers (i.e., all the points on the line) that is strictly in “size”[6] between the real numbers and the fractions (i.e., the rational numbers $0, \pm 1/2, \pm 3/5, \ldots$)? It came as a big surprise that this question, thanks to the efforts and insight of Kurt Gödel and US mathematician Paul Cohen, cannot be answered within the accepted axioms for set theory. There are models of set theory where the statement is true, and likewise models where it is false.

Hilbert's 6th problem asks for the axiomatization of physics. In the same way that Euclid had axiomatized planar geometry, one could ask if there exists an axiomatic system for nature, no less. This was in the naive days of physics: The first hints of quantum mechanics had emerged, the special theory of relativity was 5 years away. Needless to say the problem has not been solved to this day, and remains the holy grail of theoretical physics.

Hilbert's 8th problem asks for the resolution of the Riemann hypothesis and Goldfach's conjecture. Both remain unsolved, and most mathematicians would agree that the Riemann hypothesis is the biggest unsolved problem in mathematics. While the Riemann hypothesis is technical to state and understand, the Goldbach conjecture is probably the unsolved problem in mathematics that is the easiest to state to a layman: Given an even number greater than 2, prove that it can be written as a sum of two primes[7]. So simple, yet so complicated! We have $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$ and, e.g., $10774 = 7283 + 3491$.

Hilbert's 18th problem has a solution that is so simple that any layman can figure it out. However, the rigorous proof of the so-called Kepler problem was only obtained by Thomas Hales in 2000 by a computer assisted proof. The problem asks for the densest packing of equal sized spheres in three-dimensional space. The two solutions that exist are exactly the ways any kid would stack, e.g., tennis balls.

Hilbert's problems have played a dominating role in mathematics in the last century. Some problems have shaped the subsequent development; others have proved to be less relevant. However, important aspects of mathematics, for

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[6] Two sets, infinite or finite, have the same “size” if the elements of the two sets can be put in a one-to-one correspondence. Thus $\{1,2\}$ and $\{13,56\}$ have same size, different from that of $\{0,1,3\}$. The real numbers and the rational numbers have different size, and Hilbert's first problem asks if there is a set with size between these two.

[7] A prime is a natural number greater than 1 with no factors except 1 and the number itself. The first few read $2,3,5,7,11,13,\ldots$.

instance the formidable development of abstract methods, the development in differential equations and the consequences of modern computers, were completely missed by Hilbert.

With the upcoming millennium it was natural to look for a new set of problems for the present century. Mathematics had developed so much that it is impossible for one single mathematician to claim knowledge of mathematics at large. It was left to the Clay Mathematics Institute, a US institute funded by the businessman L. T. Clay and his wife, to establish a committee of prominent mathematicians to select a small number of important open problems. The committee came up with seven *Millennium Problems*, each carrying a monetary award of USD 1 million for its solution. Only one problem, the Riemann hypothesis, was also on Hilbert's list of problems. Due to the incredible development of mathematics during the last century, the problems are close to impossible to describe to laymen. Thus we will focus on only two of the problems. The first one asks for the solution of the Navier–Stokes equations that describe the motion of fluids, i.e., liquids and gases. The equations are based on the classical Newton equations. We all "know" that these equations have solutions; after all, computer simulations of the movement of water around ships and the movement of air around an aircraft, using the Navier--Stokes equations, describe the actual behavior with sufficient accuracy that we dare to fly in commercial aircrafts. However, the problem is to show mathematically that these equations do have solutions and with what properties. Closely related to this question is the fundamental question of turbulence. In spite of extensive activity on the problem, it is open in the physically important case of three space dimensions.

The other Millennium problem to be discussed here is the *Poincaré conjecture*[8]: Consider the surface of a sphere, say a tennis ball. On that surface we put a rubber band. Wherever we put it, we can contract the rubber band to a point. Now replace the tennis ball by a doughnut: Now we cannot contract a rubber band to a point for all possible positions of the rubber band without cutting the band. Thus the method with rubber band can be used to distinguish between surfaces with a hole (e.g., a doughnut) and surfaces without a hole (e.g., a tennis ball). The Poincaré conjecture asks the same question for three-dimensional surfaces. And this problem has recently been solved by the elusive Russian mathematician Grisha Perelman (1966–).

[8] For more background, see H. Holden, "Om Poincaré, Perelman og kuler" (In Norwegian), *Morgenbladet*, Oct. 13-19, 2006.

Grisha Perelman

Over a short period in 2004–05 he posted three preprints on a preprint depository on the Internet. The words “Poincaré conjecture” were not even mentioned once, the papers were condensed, difficult to understand and full of new ideas. The preprints caught the attention of workers in the area; Perelman was known to be an exceptional mathematician, and slowly the community of experts in the area have agreed that Perelman indeed has solved the problem. More complete treatments of his argument exceed 300 pages compared to the original 47 pages! Perelman is a very special person: He declined to accept a price for young mathematicians awarded by the European Mathematical Society. In 2006 he was awarded the highest international award for mathematicians younger than 40 years, the Fields Medal, which he also declined. He quit his position at the Steklov institute in St. Petersburg, Russia, and has stopped communicating with colleagues. Rumors say that he has retracted from mathematics. The procedure from the Clay Institute for awarding the 1 million dollar prize is involved. A decision is expected soon, and it would surprise mathematicians if he were to accept the prize, having refused other prizes. Hard as it is to understand for others, one just has to accept that a genius like Perelman operates in different ways than other people.

None of the other problems on the Clay list of Millennium problems have been solved, and they will most likely continue to be an important part of the development of mathematics in the third millennium. It is not even clear that we will see the solution of any other of the Clay problems in our time. But no one expected to see a solution of the Poincaré conjecture, so we just have to wait and see!