

Postdoctoral Fellow Xavier Raynaud
Department of Mathematical Sciences,
NTNU, Trondheim, Norway
raynaud@math.ntnu.no
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The shortest path between two points, Geodesics and Mechanics

In mathematics, a geodesic denotes the shortest path between two points. For all of us (and not only the mathematicians), this notion is very intuitive and the answer is clear: The shortest path between two points A and B is the straight line joining A and B . We become aware of this fact very early and it is natural for us, when moving from one location to another, to choose the trajectory which is closest to a line. Nevertheless, we do not resort to mathematics to do so and, even if a straight line is an elementary - and probably one of the oldest - mathematical object, it still is a mathematical abstraction. We choose straight line trajectories because we know that, by making any other choice, we waste time and energy. Time and energy are not mathematical objects but physical quantities and the connection with mathematics is given in the following statement: For an object moving between two points, among all possible trajectories, the straight line is the trajectory that minimizes the total kinetic energy for a given amount of time. Thus geodesics are curves that not only minimize the distance but also the energy. This concept can be generalized in different ways that we will try to explain and it will allow us to realize how fundamental geodesics are in mechanics, this branch of physics which is concerned by the study of the motion of bodies.

The connection between geodesics and mechanics is already evident in Newton's first law of motion:

Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.

Thus, a body in absence of external forces (imagine an object in space far away from any gravitational forces) moves in a uniform motion, that is, along a straight line and at a constant speed. The body moves along a geodesic (in this case, a straight line) and only an external force may divert it from this trajectory, in a way which is described by Newton's second law:

The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.

which can be rephrased in a more synthetic way as

$$\text{mass} \times \text{acceleration} = \text{external forces}$$

Here we are considering the simple and idealized case of a single point mass (all the mass is concentrated in a point). The second law of motion allows us to compute the motion of a mass point by giving its acceleration. The acceleration denotes the rate of change of the velocity while the velocity denotes the rate of change of the position. By Newton's second law, if there are no external forces, the acceleration is zero. Hence, the velocity is constant, its direction and value do not change and the point moves in a uniform motion, see Figure 1. It can be proved that, given two points and a time interval, the curve with zero acceleration is the one that minimizes the total kinetic energy it takes to move between the two points for the given time interval.

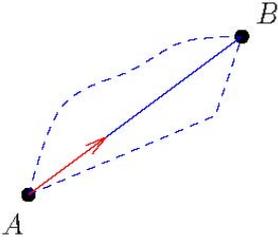
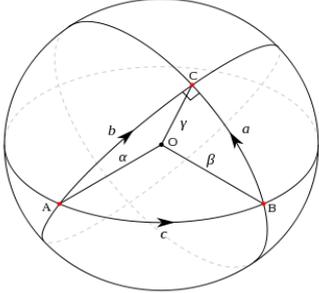


Figure 1. The point moves from A to B in a uniform motion: Its acceleration is zero so that its speed (which is represented by the red arrow) is constant. Any other trajectories (for example the ones plotted in dotted lines) yields to a higher total kinetic energy.

The word geodesic was originally used to denote the shortest path between two points on the surface of the earth. The shortest path is not given by the straight line that could be drawn on a map between the two points but by a segment of the great circle arc which contains those two points, see Figure 2. This example is important to understand that geodesics are not always straight lines. In particular as illustrated here, it depends on the geometry of the system we are looking at. A mass point, which minimizes its total kinetic energy moves along a straight line in free space or on a great circle arc if it is constrained to stay on a sphere.



(a) On a map, the geodesic between Oslo and New-York is not given by a straight line, it is a curve which corresponds to a great circle arc



(b) Plot of the three great circle arc that join the points A, B and C

Figure 2. Geodesic on earth

In the two previous cases, the kinetic energy is defined independently of the location of the point (it is equal to the product of the mass and the velocity to the square). In other cases, the location of the point matters. Let us illustrate this aspect and its consequences by looking at the following example. There is a man (denoted B in Figure 3) in the sea who is about to drown and a lifeguard (denoted A) who stands on the beach and wants to rescue him. The lifeguard is faced with the following problem: How to reach as fast as possible the man who is drowning? Going along the straight line AB is not the optimal solution in this case. The lifeguard is running much faster on the beach than he is swimming in the sea. Therefore he should try to minimize the distance he will be swimming compared to the one he will be running. One strategy is to run along a straight line to C , the point on the beach which is closest to B (see Figure 3), and then swim to B . This strategy could be better than moving in the straight line AB but still is not optimal because, if it minimizes the swimming distance, it makes the overall distance unnecessarily too large. The optimal path can be computed and the optimal angle α (see Figure 3) we obtain depends on the ratio between the velocities of the lifeguard on the beach and in the sea. The larger this ratio is, the larger this angle is.

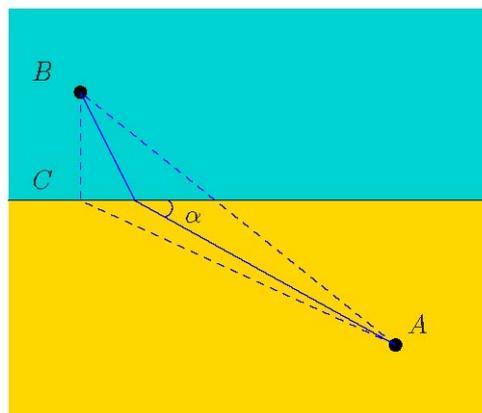
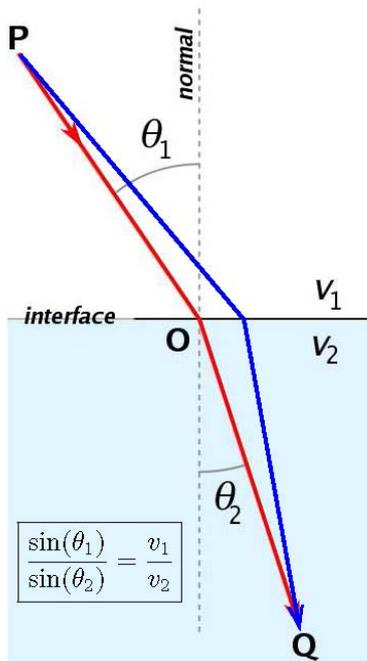
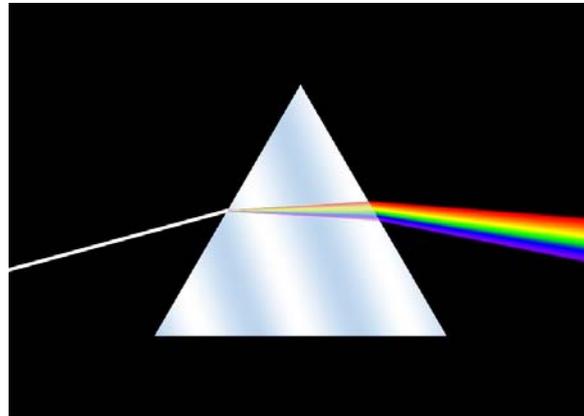


Figure 3. A lifeguard (A) is looking for the shortest path to rescue a drowning man (B). The optimal path is given by the plain line. It depends on the ratio between the speed of the lifeguard on the beach and in the water.

The same principle that we have presented in this anecdotal example apply more or less directly to some physical phenomena like diffraction. Light propagates at different speed depending on the material. For example, it goes slower in glass than in the air - as our lifeguard goes slower in water than on the sand. The path that the light follows between two points is the one that minimizes its time of travel. This principle is known as Fermat's principle or principle of least time. Given the respective speed of propagation of the light in the two materials (say air and water), one can compute the optimal path. We obtain the Snell's law which gives the angles that the light ray makes with the interface as a function of the velocities, see Figure 4a. The speed of propagation of light does not only depend on the material but also of its frequency, that is, its colour. Hence, rays of light of different colours will have different optimal path and that is why they separate when they cross an interface, as illustrated in Figure 4b.



(a) Snell's law



(b) Diffraction of light through a prism

Figure 4. Light diffraction

The optimal path for a light ray or for the lifeguard can be regarded as a geodesic problem if we modify our definition of distance. We measure the distances differently depending on the position. For example, if the lifeguard runs twice as fast as he swims, one meter in the water will count for 2 meters on the beach. By using this way of measuring distance (in mathematics, we say that we use a different metric), the optimal path reduces simply to a geodesic. It is also clear that, with this extended definition, the geodesics are in general no longer straight lines (they are broken lines in the case of a light ray).

We have now illustrated two ways in which the definition of a geodesic can be extended: By changing the geometry (from flat to curved) and by changing the metric (from a uniform to a nonuniform metric). This extended definition of geodesics enables us to consider many different mechanical systems in a unified manner. Mechanics is decomposed into two disciplines: *Kinematics*, whose aim is to describe the motion of an object without considering the cause of the motion, and *dynamics*, which studies the relationship between the motion and its cause. A *group* is a mathematical object which consists of elements and an operation acting on those elements. By definition, a group has to satisfy a set of properties (associativity, existence of a neutral element and inverses), which we do not want to detail here. An example of a group is the entire numbers (0,1,2,... and -1,-2,...), which is known to everyone, but there exist many other groups and it is fascinating to see how ubiquitous this object is in physics. In particular, groups are adequate objects to describe some very important mechanical systems (see Table 1) and, in this way, they are the adequate mathematical objects which take care of the kinematics part of our problem.

Mechanical system	Associated group
A point mass in space	group of real numbers
A rigid body	group of rotation
A fluid	group of diffeomorphism

Table 1. Mechanical systems and their associated groups

What about the dynamics? We consider two elements A and B of a group and a path that connects them. The elements A and B correspond to two different states of the mechanical system which is modelled by the group. The path provides us with a possible set of intermediate states between A and B ; it gives us a possible scenario of how the system can change from state A to state B (In Figures 1, 2 and 3, different paths have been drawn). The dynamics of the system is determined by the *least action principle*, which states that: Among all possible paths, the one that occurs is the one that minimizes the total energy, see Arnold and Khesin, 1998. The groups that are given in Table 1 are *Riemannian Lie groups*. Concretely, it means that they possess an extra structure which allows us to measure distances. We define the distance on the Riemannian Lie group such that it coincides with the kinetic energy corresponding to the mechanical system. Thus we obtain that the length of a path between A and B is precisely equal to the total kinetic energy that it takes to the system to go from A to B by following this path. It means that the geodesic then coincides with the optimal path given by the least action principle. In that way, solving a mechanical problem (finding how a system evolves) becomes equivalent to solving a geometrical problem (computing the geodesics).

Reference:

Vladimir I. Arnold and Boris A. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998