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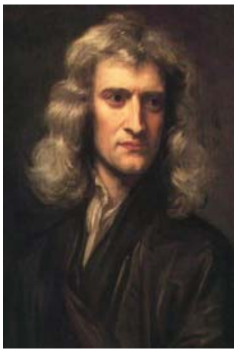
Mathematical Modeling in the World

What is the role of mathematics and mathematical modeling in the world? Can mathematics answer every question in life? When do mathematics give reliable answers? The goal of this article is not to give answers but to pose questions and to build connections and bridges between aspects and topics which appear at first sight unrelated.


Issac Newton (1643-1727)

Peter Lax one of the greatest applied mathematicians of our century referring to the relation between pure and applied mathematics during the award ceremony of the 2005 Abel Prize mentioned:

“Traditionally mathematics is divided into two kinds: pure and applied. The relation of the two is delicate. The great applied mathematician Joe Keller's definition is: *pure mathematics is a branch of applied mathematics.*

He meant that mathematics, beginning with Newton, was originally concerned with answering questions in physics, it is only later that the tools and concepts used were elaborated into theories that took on lives of their own."

What is the role of Newton as far as mathematical modeling is concerned? To answer this question one should review the fundamental laws of motion.

Newton’s laws of motion:

Dynamics is the study of motion and forces. A fundamental concept in dynamics is the dynamical system; objects under motion or under the influence of forces are called dynamical systems. For example a satellite in orbit around the Earth is a dynamical system. A beating heart might be considered a dynamical system, as well as the motion of blood for instance.

Objects only move relative to other objects, so if there is motion there is a system. By applying the laws of nature to a dynamical system, we may determine its future behavior.

It may sound very mystifying, but that is really what science is all about. Predicting the future.

The technique of predicting the “evolution” or “future” of a dynamical system by applying physical laws which govern its change as time passes is called **mathematical modeling**. The way it works is this. We discover somehow the laws relating the physical quantities of the dynamical system to time; this can be achieved either through observation or experiment. Next, we express those laws in mathematical terms. The resulting equations may involve the derivative of some of the variables with respect to time. Then we solve the equations or we show that the solution exists in a certain suitable sense and using this knowledge one can obtain the solution either analytically or numerically and even determine physically relevant properties of that solution.

This is called modeling because the mathematical functions derived from the laws of physics display the behavior of the observable quantities of the actual system.

Newton's first law of motion, also known as the law of inertia, was inspired by the work of Galileo. It states that a particle's velocity will not change unless a force is applied to the particle. This means that if a "body" (a continuum or a collection of particles having certain mass) is at rest it remains at rest unless a force is applied. If a body is moving with some velocity, it will continue to move with the same velocity unless a force is applied. The first law is really just the definition of a privileged observer, the so called *inertial frame* in which a body without force appears to persist in its uniform motion**.** Objects keep on doing what they are doing!

Newton's second law expresses a quantitative relationship between forces and motion. The velocity of a given body is defined as the rate of change of position $\mathbf{r}=\mathbf{r}(t)$, a small displacement divided by the corresponding change in time, that is $\mathbf{v} = \Delta\mathbf{r} / \Delta t$. The rate of change of velocity, in turn, is called "acceleration." The acceleration $\mathbf{a}$ of a body is the change of velocity divided by the corresponding change in time, namely $\mathbf{a} = \Delta\mathbf{v} / \Delta t$.

The second law states that the acceleration of a body is proportional to the force on it. This is consistent with our experience that the harder we push on a moveable body, the quicker its speed changes. The second law goes on to state that the constant of proportionality between the force and the acceleration is the "mass" of the body. In the form of an equation the second law reads $\mathbf{F}=m*\mathbf{a}$, where $\mathbf{F}$ is the force vector, m is the scalar mass, and $\mathbf{a}$ is the acceleration vector. The mass may be considered the property of a body that determines its resistance to changing its velocity. More mass means more force needed to accelerate.

Newton's third law addresses the nature of forces. The implicit assumption is that a force is simply a manifestation of the interaction between a pair of bodies. If you push an object it pushes back! This is why rockets fly and the reason why spacecrafts can reach the moon. The third law states that the force resulting from the interaction of two bodies acts with equal magnitude on both of them and in opposite directions. For every action, there is an equal and opposite reaction.

These three laws of motion, credited to Newton, are enough to allow us to get started in developing and analyzing mathematical models of some dynamical systems.

Based on Newton’s law of motion scientists have derived governing equations for the motion of fluids **[1]**, motion of gases which under the influence of chemical reaction lead to combustion, motion of multi-component mixtures such as astrophysical plasma, fluid-particle interaction as in the case of the motion of bacteria in a river or the blood, fluid-solid interaction, interaction of fluids with deformable bodies such as bio-membranes, motion of elastic bodies etc, aiming at investigating fundamental questions in science, engineering, biology, medicine and other applied sciences.

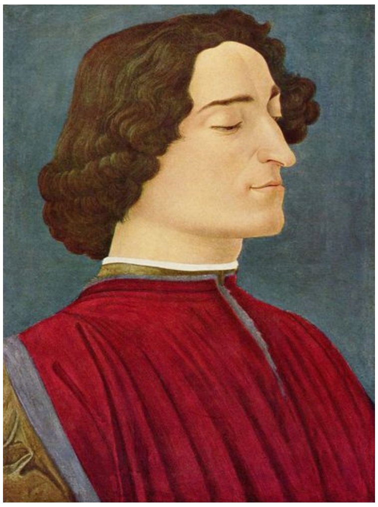
The macroscopic description, views the “body” (fluid, gas, mixture, material etc) as a continuum occupying at a given time **t** a certain domain in the physical space, and the state of the material is characterized by the so-called macroscopic variables. In the case of the Navier-Stokes system describing motion of fluids the macroscopic variables are the density, velocity vector field and the temperature field. The system involves various other quantities such as the heat flux, the viscous stress tensor, the species diffusion flux. Each one of these quantities, often referred as transport fluxes, depend on the macroscopic variables and describe a certain physical process. The pressure, the temperature and the entropy (which can be viewed as measure of the disorder in a system) are interrelated through various constitutive relations which reflect the physical properties of the material. The physical setting and the conditions surrounding the dynamical system effects the mathematical modeling in a significant way. The constitutive relations, which characterize the motion of the material or mixture take different form under different scenarios. The so-called Gibbs relation in thermodynamics describes the relationship between changes in chemical potential for components in a thermodynamical system; this relation takes different form depending on whether we are dealing with one-component fluid, or multi-component fluid mixtures **[2] [5]**, whether we are dealing with a single temperature system or a multi-temperature mixture, whether the system is assumed to be in equilibrium or not **[4]**. The physical setting, the fundamental laws of physics influence the mathematical modeling.

Are you surprised? What about art?

If you walk in any museum in the world, one can see representatives of different art movements in painting. If one compares the art work created in Venice during the period of *High Renaissance* (period between 1450 and 1527) with the art work created in Florence in the same period, one can notice some notable distinctions which can be attributed among other things to the distinct physical environment in these parts of the world.

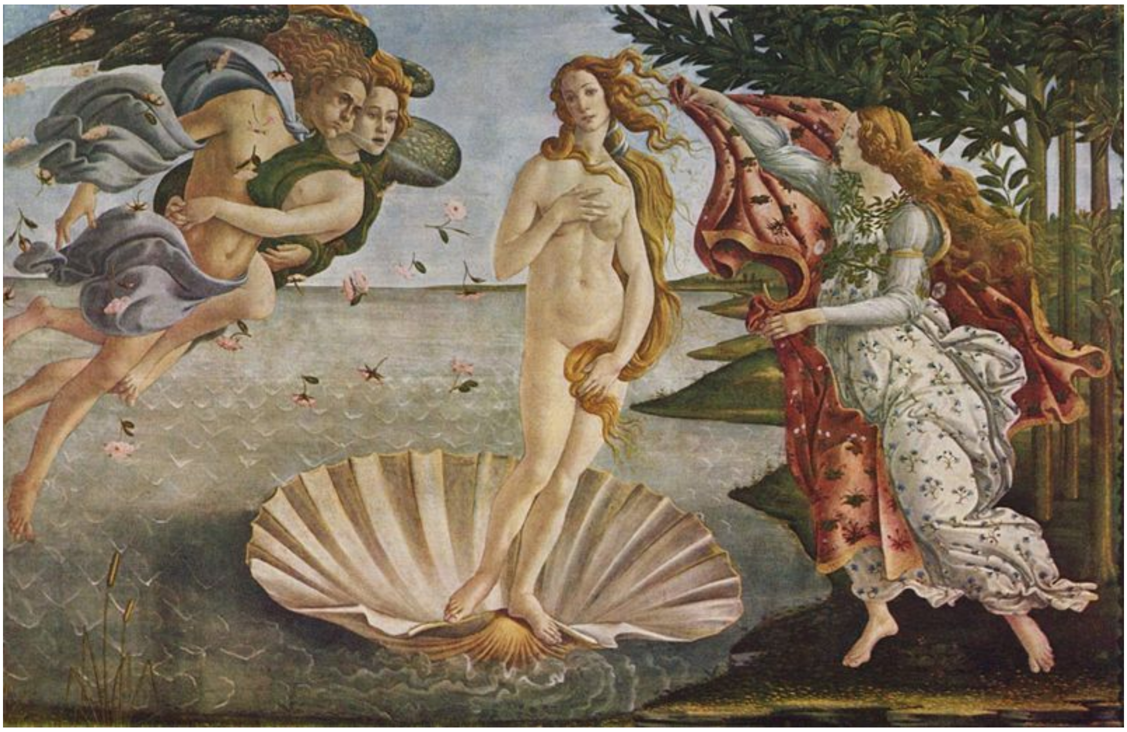
Sandro Botticelli (Florence 1445-1510)

Above you can see the self-portrait of Sandro Botticcelli (1445-1510); one representative of the art movement originated in Florence. Two of his most famous pieces are the portrait of Giuliano de' Medici (1477), and the birth of Venus (1486) presented below.



Giuliano de' Medici (1477)

The birth of Venus (1486)



In the work of Botticelli the boundary of the various objects in his paintings are very well defined. It is as if Botticelli first created a sketch with a pencil and in the sequel he filled it in with color.


Giorgione (Venice 1477-1510)

Giorgione (1477-1510) is a representative of the art movement originated in Venice. Two of his most famous pieces are the Tempest (1508) and the Sleeping Venus (1510) presented below.


The Tempest (1508)

Sleeping Venus (1510)

In the work of Giorgione one gets the impression that the boundary (of the various forms or objects) is created with the color. If you pay attention to his self-portrait above you will notice that there is no clear separation between his jacket, for instance, and the background. This is very characteristic of the art movement originated in Venice **[3]** and it is in accordance with the physical and geographical characteristics of this part of the world. The notion of boundary is a very “relaxed notion” in a region surrounded by water the location of which changes constantly.

Mathematical modeling, like art, is affected and influenced by physical principles, the setting of the problem, as we say, and it is only then, when mathematical modeling respects the laws of nature, the fundamental laws of physics (or clear experimental evidence) that it can succeed giving physically relevant and effective answers to practical problems in science and engineering.

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